

ON HIGHLY REGULAR STRONGLY REGULAR GRAPHS

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ABSTRACT. In this paper we unify many existing regularity conditions for graphs, including strong regularity, k -isoregularity, and the t -vertex condition. We develop an algebraic composition/decomposition theory of regularity conditions.

Using our theoretical results we show that a family of graphs satisfying the 7-vertex condition fulfills an even stronger condition — $(3, 7)$ -regularity (the notion is defined in the text).

Derived from this family we obtain a new infinite family of non-rank 3 strongly regular graphs satisfying the 6-vertex condition.

1. INTRODUCTION

Strongly regular graphs (srgs) are simple regular graphs with the property that the number of common neighbors of a pair of vertices depends only on whether or not the two vertices are connected by an edge. They arise, e.g., as the 2-orbit-graphs of permutation groups of rank three (such srgs are usually called *rank 3 graphs* or 2-homogeneous graphs). All rank 3 graphs are known (cf. [14, 15]). However, by no means all srgs arise in this way. In fact strongly regular graphs exist in such an abundance that a complete classification up to isomorphism is hopeless (cf. [21, 6, 16]). In order to get a better combinatorial approximation for rank 3 graphs it is necessary to impose stronger regularity conditions on strongly regular graphs. One possible such regularity-condition is the so called *t-vertex condition* that was introduced by M. D. Hestenes and D. G. Higman in [9]. An graph is said to fulfill the t -vertex condition if the number of subgraphs with $\leq t$ vertices of a given isomorphism type over a fixed pair of vertices is only depending on whether or not the vertices are connected by an edge or they are equal. Thus the t -vertex condition is in fact a class of regularity conditions parameterized by t which is generalizing the regular conditions of strongly regular graphs. In particular, the srgs are precisely the class of graphs that fulfill the 3-vertex condition.

Another class of regularity conditions strengthening strong regularity is k -isoregularity. A graph is said to be k -isoregular if for every subset S of at most k vertices the number common neighbors of the

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elements of S depends only of the isomorphism type of the subgraph induced by S . The srgs are precisely the 2-isoregular graphs. Just like the t -vertex condition is a combinatorial approximation 2-homogeneity, k -isoregularity is a combinatorial approximation k -homogeneity. The notion of k -isoregularity originates in works by J. M. J. Buczak, Ja. Goltand, and M. Klin ([1, 8]) in the course of the classification of the finite homogeneous graphs.

Every 5-isoregular finite graph is homogeneous. Similarly, it was conjectured by M. Klin (cf. [5]) that there is a number t_0 such that an srg is 2-homogeneous if and only if it satisfies the t_0 -vertex condition. In order to prove or refute this conjecture it is necessary to have good methods for deciding whether a given graph fulfills the t -vertex condition. Already in [9] Hestenes and Higman noticed that in order to verify the 4-vertex condition it is enough to test it for two types of subgraphs. More results, how to simplify the testing of the t -vertex condition appeared in [11] and [18].

In this paper we develop a general theory of regularity conditions applicable to many categories of combinatorial objects. This leads us to new criteria for the t -vertex condition and for (k, t) -regularity (a regularity condition that strengthens the concept of m -isoregularity in the same way as the t -vertex condition strengthens the concept of 2-isoregularity).

Using our theory, we show that the point graphs of partial quadrangles fulfill the 5-vertex condition. Moreover, we show that if the point graph of a partial quadrangle is 3-isoregular, then it is $(3, 7)$ -regular. In particular, the point graphs of generalized quadrangles of order (q, q^2) are $(3, 7)$ -regular. As a consequence, we obtain that the point graphs of partial quadrangles of order $(q - 1, q^2, q^2 - q)$ satisfy the 6-vertex condition.

2. PRELIMINARIES

A graph is a pair (V, E) where V is a finite set of vertices and $E \subseteq \binom{V}{2}$ is a set of undirected edges.

If Γ is a graph, then by $V(\Gamma)$ we denote the vertex set and by $E(\Gamma)$ we denote the edge-set of Γ . If $M \subseteq V(\Gamma)$, then by $\Gamma(M)$ we denote the subgraph of Γ induced by M .

As usual, the *order* of a graph is the number of its vertices and the *valency* of a vertex is the number of its neighbors.

A graph Γ for which $E(\Gamma) = \binom{V(\Gamma)}{2}$ is called *complete graph*. A complete graph of order n will be denoted by K_n . The *complement* of a graph Γ is $(V(\Gamma), \binom{V(\Gamma)}{2} \setminus E(\Gamma))$. It is denoted by $\bar{\Gamma}$.

The class of all graphs can be naturally equipped with a concept of homomorphisms: A *graph-homomorphism* (or short: *homomorphism*) from a graph Γ_1 to a graph Γ_2 is a function $f : V(\Gamma_1) \rightarrow V(\Gamma_2)$ with the property that for each $\{v, w\} \in E(\Gamma_1)$ we have that $\{f(v), f(w)\} \in E(\Gamma_2)$.

$E(\Gamma_2)$. Clearly, with this concept of homomorphisms the class of graphs forms a category that we will denote by **Graph**.

An injective homomorphism $f : \Gamma_1 \rightarrow \Gamma_2$ is called *embedding* if for all $\{v, w\} \in \binom{V(\Gamma_1)}{2} : \{v, w\} \in E(\Gamma_1) \iff \{f(v), f(w)\} \in E(\Gamma_2)$. Bijective embeddings are called *isomorphisms*.

Definition 1. Let $\Delta, \Theta_1, \Theta_2$ be graphs and let $f_1 : \Delta \rightarrow \Theta_1, f_2 : \Delta \rightarrow \Theta_2$ be homomorphisms. A compatible cocone of (f_1, f_2) is a pair (g_1, g_2) such that $g_1 : \Theta_1 \rightarrow \Theta, g_2 : \Theta_2 \rightarrow \Theta$ for some graph Θ , such that the following diagram commutes:

$$(1) \quad \begin{array}{ccc} \Theta_1 & \xrightarrow{g_1} & \Theta \\ f_1 \uparrow & & \uparrow g_2 \\ \Delta & \xrightarrow{f_2} & \Theta_2 \end{array}$$

(g_1, g_2) is called *limiting cocone* of (f_1, f_2) if for any other compatible cocone (h_1, h_2) of (f_1, f_2) where $h_1 : \Theta_1 \rightarrow \Gamma, h_2 : \Theta_2 \rightarrow \Gamma$ there exists a unique homomorphism $k : \Theta \rightarrow \Gamma$ such that the following diagram commutes:

$$\begin{array}{ccc} & & \Gamma \\ & \nearrow h_1 & \\ \Theta_1 & \xrightarrow{g_1} & \Theta \\ f_1 \uparrow & & \uparrow g_2 \\ \Delta & \xrightarrow{f_2} & \Theta_2 \end{array} \quad \begin{array}{c} \nearrow h \\ \searrow h_2 \end{array}$$

In that case the diagram (1) is called a *pushout-square*.

For us, only the special case when (f_1, f_2) is a pair of embeddings will be of interest. In this case, for every limiting cocone (g_1, g_2) of (f_1, f_2) we know that g_1, g_2 are embeddings, too. A concrete construction of limiting cocones of pairs of embeddings in the category **graph** goes as follows:

Construction. Let $f_1 : \Delta \hookrightarrow \Theta_1, f_2 : \Delta \hookrightarrow \Theta_2$ be embeddings. Let $\tilde{\Theta}$ be the disjoint union of Θ_1 and Θ_2 . Let $\theta \subseteq V(\tilde{\Theta})^2$ be the smallest equivalence relation that contains

$$\{(f_1(v), f_2(v)) \mid v \in V(\Delta)\}.$$

Let $\Theta := \tilde{\Theta}/\theta$. Finally, let $g_1 : \Theta_1 \hookrightarrow \Theta$ and $g_2 : \Theta_2 \hookrightarrow \Theta$ be given by

$$g_1 : v \mapsto [c]_\theta, \quad g_2 : w \mapsto [w]_\theta.$$

Then (g_1, g_2) is a limiting cocone for (f_1, f_2) .

Note that θ has equivalence classes of size ≤ 2 . One can imagine that Θ is obtained by glueing Θ_1 and Θ_2 together at a copy of Δ , which is marked in Θ_1 and Θ_2 through f_1 and f_2 , respectively.

3. GRAPH-TYPES AND REGULARITY-CONDITIONS

The t -vertex condition arises from a local invariant of pairs of vertices of a graph. Let $\Gamma = (V, E)$ be a graph and let $(x, y) \in V^2$. We consider all induced subgraphs of Γ that contain x and y and that have order $\leq t$. Two such subgraphs are said to be of the same type if they are isomorphic by an isomorphism that fixes x and y . The possible types of subgraphs correspond to all isomorphism classes of graphs of order $\leq t$ with a pair of distinguished vertices. Now to the pair (x, y) we may associate a function $\varphi_{x,y}$ from the types to the natural numbers that maps every type to the number of induced subgraphs of Γ that contain x and y and that are of this type. Graphs where the function $\varphi_{x,y}$ does not depend directly on the pair (x, y) but only on whether $x = y$ or $\{x, y\} \in E$ or $\{x, y\} \in \binom{V}{2} \setminus E$, are said to fulfill the t -vertex condition. In the following we will give an equivalent definition of the t -vertex condition using the language of category-theory.

In the previous section we defined the category **Graph**. In order to characterize the t -vertex condition and other invariants of graphs we will consider another category, derived from **Graph**: We will in the sequel work with graphs with a distinguished subgraph.

Definition 2. A graph-type \mathbb{T} is a triple (Δ, τ, Θ) where Δ and Θ are graphs and $\tau : \Delta \hookrightarrow \Theta$ is an embedding. The *order* of \mathbb{T} is the pair (n, m) where n is the order of Δ and m is the order of Θ . The graph Δ is called *basis* of \mathbb{T} .

Graph types form a category in a natural way. Namely, for given graph-types $\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$ and $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$ a morphism from \mathbb{T}_1 to \mathbb{T}_2 is pair (f, g) of graph-homomorphisms such that $f : \Delta_1 \rightarrow \Delta_2$, $g : \Theta_1 \rightarrow \Theta_2$ and such that the following diagram commutes.

$$\begin{array}{ccc} \Delta_1 & \xrightarrow{\iota_1} & \Theta_1 \\ f \downarrow & & \downarrow g \\ \Delta_2 & \xrightarrow{\iota_2} & \Theta_2 \end{array}$$

Lemma 3. *Given natural numbers m , such that $m \leq n$. Then there are just finitely many isomorphism classes of graph-types of order (m, n) .*

Proof. There are just finitely many (say, k) unlabeled graphs of order n . Moreover, every graph of order n accounts for at most $\binom{n}{m}$ graph-types of order (m, n) . Hence, there are at most $k \binom{n}{m}$ isomorphism classes of graph-types of order (m, n) . \square

Definition 4. Let $\mathbb{T} = (\Delta, \iota, \Theta)$ be a graph type, let Γ be a graph, and let $\kappa : \Delta \hookrightarrow \Gamma$ be an embedding. We say that an embedding $\hat{\kappa} : \Theta \hookrightarrow \Gamma$

extends κ along ι if the following diagram commutes:

$$\begin{array}{ccc} \Theta & \xrightarrow{\hat{\kappa}} & \Gamma \\ \uparrow \iota & \nearrow \kappa & \\ \Delta & & \end{array}$$

The number of all extensions of κ along ι will be denoted by $\#(\Gamma, \mathbb{T}, \kappa)$. If $\#(\Gamma, \mathbb{T}, \kappa)$ does not depend on κ , then this number will be denoted by $\#(\Gamma, \mathbb{T})$. In this case Γ will be called \mathbb{T} -regular.

Remark. Let Γ be a graph. Consider fixed graph-types \mathbb{T}_1 , and \mathbb{T}_2 . Suppose that $\mathbb{T}_1 \cong \mathbb{T}_2$. If Γ is \mathbb{T}_1 -regular then it is also \mathbb{T}_2 -regular. Moreover, in this case we have $\#(\Gamma, \mathbb{T}_1) = \#(\Gamma, \mathbb{T}_2)$.

Remark. A concept equivalent to \mathbb{T} -regularity, but in the category of complete colored graphs, was introduced and studied in [4] in relation with the t -vertex condition for association schemes.

Definition 5. We say that a graph Γ is

- $[m, n]$ -regular if it is \mathbb{T} -regular for all graph-types \mathbb{T} of order (m, n) .
- $[m, n)$ -regular if it is $[m, l]$ -regular for all $m \leq l \leq n$,
- $(m, n]$ -regular if it is $[k, n]$ -regular for all $k \leq m$,
- (m, n) -regular if it is $[k, n)$ -regular for all $k \leq m$,

(m, n) -regularity is a combinatorial approximation of the notion of m -homogeneity. Recall:

Definition 6. A graph Γ is called *m -homogeneous* if every isomorphism between induced subgraphs of order at most m extends to an automorphism of Γ . It is called *homogeneous* if every isomorphism between finite induced subgraphs extends to an automorphism.

Obviously, for every graph Γ we have that m -homogeneity implies (m, n) -regularity for all $m \leq n$. Moreover, if n is the order of Γ , then the (m, n) -regularity of Γ implies the m -homogeneity.

Lemma 7. A graph Γ satisfies the t -vertex condition if and only if it is $(2, t)$ -regular. \square

Let us now see how we may compose new graph types from given ones:

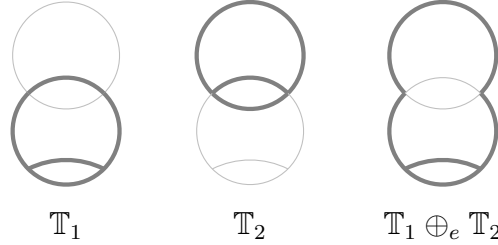
Definition 8. Given graph-types $\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$ and $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$, $e : \Delta_2 \hookrightarrow \Theta_1$.

Let Λ be a graph, $\lambda_1 : \Theta_1 \hookrightarrow \Lambda$, $\lambda_2 : \Theta_2 \hookrightarrow \Lambda$ such that the following is a pushout square:

$$(2) \quad \begin{array}{ccc} \Theta_2 & \xrightarrow{\lambda_2} & \Lambda \\ \uparrow \iota_2 & & \uparrow \lambda_1 \\ \Delta_2 & \xrightarrow{e} & \Theta_1 \end{array}$$

Then the graph-type $(\Delta_1, \lambda_1 \circ \iota_1, \Lambda)$ is called the *free sum of \mathbb{T}_1 and \mathbb{T}_2 with respect to e* . It will be denoted by $\mathbb{T}_1 \oplus_e \mathbb{T}_2$.

Remark. The following picture illustrates the construction of a free sum of types:



Definition 9. Let \mathbb{T}, \mathbb{T}_2 be graph-types. We say that \mathbb{T} is \mathbb{T}_2 -reducible, if $\mathbb{T} \cong \mathbb{T}_1 \oplus_e \mathbb{T}_2$ for some $\mathbb{T}_1 \not\cong \mathbb{T}$.

Remark. With the notions from above $\mathbb{T} = (\Delta, \iota, \Theta)$ is \mathbb{T}_2 -reducible if and only if the set $V(\Theta)$ can be decomposed as a disjoint union of subsets M_1 , M_2 , and M_3 , such that

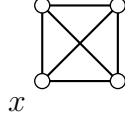
- (1) $\text{im}(\iota) \subseteq M_1 \cup M_3$,
- (2) $M_2, M_3 \neq \emptyset$
- (3) there are no edges in Θ between vertices from M_1 and vertices from M_2 ,
- (4) $\mathbb{T}'_2 := (\Theta(M_3), \iota', \Theta(M_2 \cup M_3)) \cong \mathbb{T}_2$ (here ι' denotes the identical embedding).

In this case, we have $\mathbb{T}_1 = (\Delta, \iota, \Theta(M_1 \cup M_3))$ and $\mathbb{T} \cong \mathbb{T}_1 \oplus_e \mathbb{T}'_2$, where e is the identical embedding of M_3 into $M_1 \cup M_3$.

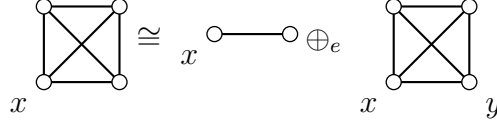
Definition 10. A graph-type \mathbb{T} is called (m, n) -irreducible if whenever $\mathbb{T} \cong \mathbb{T}_1 \oplus_e \mathbb{T}_2$ for graph types \mathbb{T}_1 and a graph-type \mathbb{T}_2 of order (k, l) with $k \leq m$ and $l \leq n$, then we already have $\mathbb{T} \cong \mathbb{T}_1$ or $\mathbb{T} \cong \mathbb{T}_2$. Otherwise, we call \mathbb{T} (m, n) -reducible

Remark. A graph-type \mathbb{T} is (m, n) -reducible if and only if it is \mathbb{T}' -reducible, for some graph-type \mathbb{T}' of order (k, l) , where $k \leq m$ and $l \leq n$, such that $\mathbb{T}' \not\cong \mathbb{T}$.

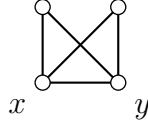
Example 11. Note that the $(1, 4)$ -type



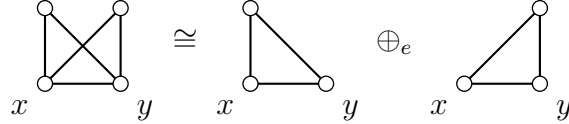
is $(2, 4)$ -reducible because we have



Moreover, the $(2, 4)$ -type



is $(2, 3)$ -reducible (and hence also $(2, 4)$ -reducible) because



Definition 12. Let $\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$, $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$ be graph-types. Then we define $\mathbb{T}_1 \preceq \mathbb{T}_2$ (\mathbb{T}_2 *dominates* \mathbb{T}_1) if there exists a morphism $(f, g) : \mathbb{T}_2 \rightarrow \mathbb{T}_1$ such that $f : \Delta_2 \rightarrow \Delta_1$ is an isomorphism and such that $g : \Theta_2 \rightarrow \Theta_1$ is surjective. If g is not an isomorphism, then we also write $\mathbb{T}_1 \prec \mathbb{T}_2$.

Lemma 13. *The predicate \preceq defines a quasi-order on graph-types. For finite graph-types $\mathbb{T}_1, \mathbb{T}_2$ we have $\mathbb{T}_1 \cong \mathbb{T}_2$ if and only if $\mathbb{T}_1 \preceq \mathbb{T}_2$ and $\mathbb{T}_2 \preceq \mathbb{T}_1$. Up to isomorphism, every finite graph-type dominates only finitely many other graph-types.*

Proof. It is clear that \preceq is reflexive and transitive (in other words, it is a quasi-order). Since we talk about finite graph types, from $\mathbb{T}_1 \preceq \mathbb{T}_2$ and $\mathbb{T}_1 \preceq \mathbb{T}_2$ follows $\mathbb{T}_1 \cong \mathbb{T}_2$.

Suppose now that $\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$, $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$ are graph-types such that $\mathbb{T}_1 \preceq \mathbb{T}_2$. Then we have $|\Theta_1| \leq |\Theta_2| =: n$. Let $m := |\Delta_2| = |\Delta_1|$. Then, by Lemma 3, there are up to isomorphism just finitely many isomorphism classes of graph-types of order (k, l) where $k \leq m$, $l \leq n$, $k \leq l$. \square

Lemma 14 (Type-counting-lemma). *Let Γ be a graph, let $\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$, $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$ be finite graph types, and let $e : \Delta_2 \hookrightarrow \Theta_1$ be an embedding.*

Then Γ is $\mathbb{T}_1 \oplus_e \mathbb{T}_2$ -regular if

- (1) Γ is \mathbb{T}_1 -regular,

- (2) Γ is \mathbb{T}_2 -regular,
- (3) Γ is \mathbb{T} -regular for every $\mathbb{T} \prec \mathbb{T}_1 \oplus_e \mathbb{T}_2$.

Proof. Suppose $\mathbb{T}_1 \oplus_e \mathbb{T}_2 = (\Delta, \iota, \Theta)$. Let λ_1, λ_2 be given such that the following is a pushout square:

$$(3) \quad \begin{array}{ccc} \Theta_2 & \xrightarrow{\lambda_2} & \Theta \\ \iota_2 \uparrow & & \uparrow \lambda_1 \\ \Delta_2 & \xrightarrow{e} & \Theta_1 \end{array}$$

and such that $\iota = \lambda_1 \circ \iota_1$.

Let us fix one embedding $\kappa : \Delta_1 \rightarrow \Gamma$. Our goal it is, to count all extensions of κ along ι .

Suppose $\hat{\kappa} : \Theta \hookrightarrow \Gamma$ extends κ along ι (i.e., $\kappa = \hat{\kappa} \circ \iota$). Define $\mu_1 : \Theta_1 \rightarrow \Gamma$ by $\mu_1 := \hat{\kappa} \circ \lambda_1$ and $\mu_2 : \Theta_2 \hookrightarrow \Gamma$ by $\mu_2 := \hat{\kappa} \circ \lambda_2$. Then, by construction, we have $\mu_1 \circ e = \mu_2 \circ \iota_2$. In other words, (μ_1, μ_2) forms a compatible cocone for (e, ι_2) . And $\hat{\kappa}$ is the unique mediating morphism from the limiting cocone (λ_1, λ_2) to (μ_1, μ_2) . Note also that μ_1 extends κ along ι_1 , by construction.

Thus, to every extension of κ along ι corresponds a unique compatible cocone (μ_1, μ_2) for (e, ι_2) such that μ_2 extends κ along ι_1 (such cocones will be called κ -compatible cocones of (e, ι_2) into Γ). In the following we count all those κ -compatible cocones of (e, ι_2) into Γ that define extensions of κ along ι :

Let $\mu_1 : \Theta_1 \hookrightarrow \Gamma$ and $\mu_2 : \Theta_2 \hookrightarrow \Gamma$ be given such that (μ_1, μ_2) is a κ -compatible cocone for (e, ι_2) into Γ . Then, by the universal property of pushouts, there exists a unique homomorphism $h : \Theta \rightarrow \Gamma$ such that the following diagram commutes:

$$\begin{array}{ccc} & & \Gamma \\ & \mu_2 \nearrow & \uparrow \\ \Theta_2 & \xrightarrow{\lambda_2} & \Theta \\ \iota_2 \uparrow & & \uparrow \lambda_1 \\ \Delta_2 & \xrightarrow{e} & \Theta_1 \end{array} \quad \begin{array}{c} \nearrow h \\ \searrow \mu_1 \end{array}$$

Let $\tilde{\Theta}$ be the subgraph of Γ induced by the image of h . Then $\tilde{\mathbb{T}} = (\Delta_1, h \circ \iota, \tilde{\Theta})$ is a graph-type since $h \circ \iota = h \circ \lambda_1 \circ \iota_1 = \mu_1 \circ \iota_1$. Moreover, $\tilde{\mathbb{T}} \preceq \mathbb{T}$, because $(1_{\Delta_1}, h) : \mathbb{T} \rightarrow \tilde{\mathbb{T}}$, and since $h : \Theta \rightarrow \tilde{\Theta}$ is surjective. Finally, the identical embedding from $\tilde{\Theta}$ to Γ extends κ along $h \circ \iota$. On the other hand, any extension $\hat{\kappa}$ of κ along $h \circ \iota$ defines a unique compatible cocone $(\hat{\mu}_1, \hat{\mu}_2)$ of (e, ι_1) according to $\hat{\mu}_1 = \hat{\kappa} \circ h \circ \lambda_1$ and $\mu_2 = \hat{\kappa} \circ h \circ \lambda_2$, and $\hat{\kappa} \circ h$ is the mediating morphism.

In this way, every κ -compatible cocone (μ_1, μ_2) of (e, ι_2) into Γ defines a graph-type below \mathbb{T} . By Lemma 13, only finitely many types are defined by compatible cocones, up to isomorphism. Let $\tilde{\mathbb{T}}_1, \dots, \tilde{\mathbb{T}}_k$ be a transversal of isomorphism types of graph-types defined by compatible cocones of (e, ι_2) into Γ (here $\tilde{\mathbb{T}}_i = (\Delta_1, \tilde{\iota}_i, \tilde{\Theta}_i)$, for $i \in \{1, \dots, k\}$). Without loss of generality, suppose that $\mathbb{T}_1 = \mathbb{T}_1 \oplus_e \mathbb{T}_2$.

Now we observe that there is a one-to-one correspondence between κ -compatible cocones of (e, ι_2) into Γ and pairs $(i, \hat{\kappa})$ such that $i \in \{1, \dots, k\}$ and such that $\hat{\kappa}$ extends κ along $\tilde{\iota}_i$. The number of κ -compatible cocones of (e, ι_2) into Γ is equal to $\#(\Gamma, \mathbb{T}_1) \cdot \#(\Gamma, \mathbb{T}_2)$. Thus, we obtain

$$\#(\Gamma, \mathbb{T}_1) \cdot \#(\Gamma, \mathbb{T}_2) = \sum_{i=1}^k \#(\Gamma, \tilde{\mathbb{T}}_i).$$

Finally, we compute

$$\#(\Gamma, \mathbb{T}_1 \oplus_e \mathbb{T}_2) = \#(\Gamma, \mathbb{T}_1) \cdot \#(\Gamma, \mathbb{T}_2) - \sum_{i=2}^k \#(\Gamma, \tilde{\mathbb{T}}_i).$$

Thus, Γ is $\mathbb{T}_1 \oplus_e \mathbb{T}_2$ -regular. \square

The proofs of the following propositions will make use of the following (very basic), induction principle for finite posets.

Lemma 15. *Let (P, \leq) be a partially ordered set and let $B \subseteq P$. If*

$$(4) \quad \forall p \in P : (\{q \in P \mid q < p\} \subseteq B \Rightarrow p \in B),$$

then we already have that B is equal to P .

Proof. Suppose that (4) holds for B , but that $B \neq P$. Let x be a minimal element of $P \setminus B$ in (P, \leq) (this exists because P is finite). Then for all $y \leq x$ we have $y \in B$. Thus, by (4), we also have $x \in B$ — contradiction. \square

Proposition 16. *Let Γ be an (m, m) -regular graph. Then, Γ is (m, n) -regular if and only if it is $[m, n]$ -regular.*

Proof. Clearly, from (m, n) -regularity follows $[m, n]$ -regularity.

So suppose that Γ is $[m, n]$ -regular and (m, m) -regular. Let P be a transversal of the isomorphism classes of graph-types of order (k, l) for $k \leq m$ and for $l \leq n$. Then, by Lemma 3, (P, \preceq) is a finite poset. Moreover, whenever $\mathbb{T} \in P$ and $\mathbb{T}' \preceq \mathbb{T}$, then \mathbb{T}' is isomorphic to an element of P .

Let $\mathbb{T} = (\Delta, \iota, \Theta) \in P$ be of order (k, l) . Suppose also that for all $\mathbb{T}' \prec \mathbb{T}$ we have that Γ is \mathbb{T}' -regular. If $l \leq m$, then Γ is \mathbb{T} -regular, by assumption. So suppose that $m < l \leq n$. Let $\hat{\Delta}$ be an induced subgraph of order m of Θ that contains the image of ι , and let $\hat{\iota}$ be

the identical embedding of $\hat{\Delta}$ into Θ . Then $\mathbb{T}_1 := (\Delta, \iota, \hat{\Delta})$ is a graph-type of order (k, m) , and $\mathbb{T}_2 := (\hat{\Delta}, \hat{\iota}, \Theta)$ is a graph-type of order (m, l) . Moreover, $\mathbb{T} \cong \mathbb{T}_1 \oplus_{\hat{\iota}} \mathbb{T}_2$. By the assumptions, we have that Γ is \mathbb{T}_1 - and \mathbb{T}_2 -regular. Hence, by the graph-counting lemma, we conclude that Γ is \mathbb{T} -regular.

By the arguments above and by Lemma 15, Γ is \mathbb{T} -regular for all graph-types \mathbb{T} from P . In other words, Γ is (m, n) -regular. \square

Note that a graph is $(2, 2)$ -regular if and only if it is regular. Thus, the previous proposition generalizes a well-known result by A.V. Ivanov:

Theorem 17 (A.V. Ivanov [11, Prop.2.1]). *Let Γ be a regular graph. Then Γ satisfies the t -vertex condition if and only if it is $[2, t)$ -regular.* \square

Definition 18. A graph $\Gamma = (V, E)$ is called *k -isoregular* if for every subset $X \subseteq V$ with $|X| \leq k$ the number of vertices $v \notin X$ that are adjacent to every element of X does not depend on X but only the isomorphism type of the subgraph of Γ that is induced by X .

Proposition 19. *Let Γ be a graph and let $k > 0$ be a natural number. Then the following are equivalent:*

- (1) Γ is k -isoregular,
- (2) Γ is $[l, l + 1]$ -regular for every $1 \leq l \leq k$,
- (3) Γ is $(k, k + 1)$ -regular.

Proof. “(1) \Rightarrow (2):” Let $l \in \{1, \dots, k\}$, and let P be a transversal of the isomorphism classes of graph-types of order (l, m) where $l \leq m \leq l + 1$. Then, by Lemma 3, (P, \preceq) is a finite poset. Moreover, since $\mathbb{T}' \prec \mathbb{T}$ implies that the order of \mathbb{T}' is (l, m) for some $l \leq m \leq l + 1$, for every $\mathbb{T}' \prec \mathbb{T}$ there exists a unique $\mathbb{T}'' \in P$ such that $\mathbb{T}' \cong \mathbb{T}''$.

In the following we will show that Γ is \mathbb{T} -regular, for all $\mathbb{T} \in P$. Let $\mathbb{T} = (\Delta, \iota, \Theta)$ be an element of P . Moreover, suppose that for all $\mathbb{T}' \prec \mathbb{T}$ the graph Γ is \mathbb{T}' -regular. If the order of \mathbb{T} is (l, l) , then Γ is \mathbb{T} -regular. So suppose that \mathbb{T} has order $(l, l + 1)$. Let v be the unique vertex of Θ that is not in the image of ι . If v has valency l , then Γ is \mathbb{T} -regular, because it is k -isoregular. So, suppose finally that the valency of v is equal to $m < l$. Let $\hat{\Delta}$ be the subgraph of Θ induced by the neighbors of v , let $\hat{\Theta}$ be the subgraph of Θ induced by the neighbors of v , together with v itself, and let $\hat{\iota}$ be the identical embedding of $\hat{\Delta}$ into $\hat{\Theta}$. Let finally Θ' be the subgraph of Θ induced by the image of ι . Then $\mathbb{T}_1 := (\Delta, \iota, \Theta')$ and $\mathbb{T}_2 := (\hat{\Delta}, \hat{\iota}, \hat{\Theta})$ are graph-types. Moreover, $\mathbb{T} \cong \mathbb{T}_1 \oplus_{\hat{\iota}} \mathbb{T}_2$. Then \mathbb{T}_1 is of order (l, l) thus, Γ is \mathbb{T}_1 -regular. Moreover, \mathbb{T}_2 is of order $(m, m + 1)$ and the \mathbb{T}_2 -regularity of Γ follows from the k -isoregularity of Γ . Now, from the type-counting lemma it follows that Γ is \mathbb{T} -regular. Finally, from Lemma 15 it follows that Γ is regular for all types from P . In particular, Γ is $[l, l + 1]$ -regular.

“(2) \Rightarrow (3):” We show that Γ is $(l, l+1)$ -regular for all $l \in \{1, \dots, k\}$. We proceed by induction on l . Clearly, Γ is $(1, 2)$ regular if and only if it is $[1, 1]$ -, $[1, 2]$, and $[2, 2]$ -regular. The first and the last regularity condition are trivially fulfilled and the $[1, 2]$ regularity is given by assumption. Suppose, we know that Γ is $(l, l+1)$ -regular and $[l, l+1]$ -regular. Then from the $(l, l+1)$ -regularity follows immediately the (l, l) -regularity. Moreover, we have that Γ is $[l, l+1]$ -regular, because Γ is $[l, l]$ -regular and Γ is $[l, l+1]$ -regular. Hence, by Proposition 16, it follows that Γ is $(l, l+1)$ -regular. Ultimately we have that Γ is $(k, k+1)$ -regular.

“(3) \Rightarrow (1):” k -isoregularity of Γ follows immediately from the $(k, k+1)$ -regularity. \square

S. Reichard in [18] characterized, when a k -isoregular graph satisfies the t -vertex condition:

Theorem 20 (S. Reichard [18]). *Let Γ be a k -isoregular graph that satisfies the $(t-1)$ -vertex condition for $t > 3$. Then, in order to verify the t -vertex condition it suffices to test the \mathbb{T} -regularity for graph-types $\mathbb{T} = (\Delta_1, \Delta_2, \iota)$ of order $(2, y)$ with $y \leq t$ with the property that all vertices of Δ_2 that are not in the image of ι have valency $\geq k+1$.*

The following proposition generalizes Theorem 20:

Proposition 21. *Let Γ an (m, t) -regular graph. Let \mathcal{M} be a set of graph-types and suppose that Γ is \mathbb{T} -regular, for all $\mathbb{T} \in \mathcal{M}$. Then, in order to verify the $(m, t+1)$ -regularity of Γ it suffices to test the \mathbb{T} -regularity for graph-types of order (m, l) with $l \leq t+1$ that are \mathbb{T} -irreducible for all $\mathbb{T} \in \mathcal{M}$.*

Proof. Let P be a transversal of the isomorphism classes of graph-types of order (m, l) for all $l \leq t+1$. Then, by Lemma 3, (P, \preceq) is a finite poset. Moreover, whenever $\mathbb{T} \in P$ and $\mathbb{T}' \preceq \mathbb{T}$, then \mathbb{T}' is isomorphic to an element of P .

Let $\mathbb{T} \in P$ and suppose that Γ is \mathbb{T}' -regular for all $\mathbb{T}' \prec \mathbb{T}$. If \mathbb{T} is of order (m, n) for $n \leq t$, then Γ is \mathbb{T} -regular, by assumption. So assume that $\mathbb{T} = (\Delta, \iota, \Theta)$ is of order $(m, t+1)$. If \mathbb{T} is $\widehat{\mathbb{T}}$ -irreducible for all $\widehat{\mathbb{T}} \in \mathcal{M}$, then Γ is \mathbb{T} -regular, by assumption. So suppose that there exists a $\widehat{\mathbb{T}} \in \mathcal{M}$, such that \mathbb{T} is $\widehat{\mathbb{T}}$ -reducible. Then $\mathbb{T} \cong \mathbb{T}_1 \oplus_e \widehat{\mathbb{T}}$ for some graph-type $\mathbb{T}_1 \not\cong \mathbb{T}$. But then the order of \mathbb{T}_1 is (m, l) , for some $l < t+1$. Hence, by assumption \mathbb{T} is \mathbb{T}_1 -regular and $\widehat{\mathbb{T}}$ -regular. By the type-counting lemma we obtain that Γ is \mathbb{T} -regular.

Now, it remains to invoke Lemma 15, to obtain that Γ is regular for all types from P . Consequently, Γ is $[m, t+1)$ -regular. By assumption, Γ is (m, m) -regular. Hence, by Proposition 16, we have that Γ is $(m, t+1)$ -regular. \square

Proposition 22. *Let Γ be a graph. Then Γ is $(m, n+1)$ -regular if and only if Γ is (m, n) -regular and it is \mathbb{T} -regular for every $(m, n+1)$ -irreducible graph type \mathbb{T} of order $(m, n+1)$.*

Proof. “ \Rightarrow ” clear.

“ \Leftarrow ” Let \mathcal{M} be a transversal of the isomorphism classes of graph-types of order (k, l) , where $k \leq m$ and where $l \leq n$. By Assumption, Γ is regular for all graph-types from \mathcal{M} . Thus, by Proposition 21, in order to show that Γ is $(m, n+1)$ -regular it suffices to show that Γ is \mathbb{T} -regular, for all graph types \mathbb{T} of order (m, l) with $l \leq n+1$ that are \mathbb{T}' -irreducible, for all $\mathbb{T}' \in \mathcal{M}$.

We claim that a graph-type \mathbb{T} of order $(m, n+1)$ is $(m, n+1)$ -reducible if and only if it is \mathbb{T}' -reducible for some $\mathbb{T}' \in \mathcal{M}$.

So suppose that \mathbb{T} is $(m, n+1)$ -reducible. Then $\mathbb{T} \cong \mathbb{T}_1 \oplus_e \mathbb{T}_2$ such that \mathbb{T}_2 has order (k, k') where $k \leq m$ and $k' \leq n+1$, and where neither \mathbb{T}_1 nor \mathbb{T}_2 is isomorphic to \mathbb{T} . It follows that \mathbb{T} is \mathbb{T}_2 -reducible it remains to show that \mathbb{T}_2 is an element of \mathcal{M} . Note that \mathbb{T}_1 has order (m, k'') , where $k'' \leq n$, for otherwise $\mathbb{T}_1 \cong \mathbb{T}$. Thus, by assumption, Γ is \mathbb{T}_1 -regular. Suppose $k' = n+1$, then $k'' = k$ because $k' - k + k'' = n+1$. Since $k \leq m$ and $k'' \geq m$, we conclude $k'' = k = m$. Thus \mathbb{T}_2 has the same order like \mathbb{T} . It follows that \mathbb{T}_2 and \mathbb{T} are isomorphic—contradiction. Hence $k' \leq n$. Thus, $\mathbb{T}_2 \in \mathcal{M}$.

Suppose now that \mathbb{T} is \mathbb{T}' -reducible for some $\mathbb{T}' \in \mathcal{M}$. Then $\mathbb{T} \cong \mathbb{T}_1 \oplus_e \mathbb{T}'$, where $\mathbb{T}_1 \not\cong \mathbb{T}$. Suppose that the order of \mathbb{T}' is (k, l) . Then $k \leq m$ and $l \leq n$. It follows that $\mathbb{T} \not\cong \mathbb{T}'$. From the definition of $(m, n+1)$ -irreducibility, it follows that \mathbb{T} is $(m, n+1)$ -reducible.

This finishes the proof. \square

Definition 23. Let $\mathbb{T} = (\Delta, \iota, \Theta)$ be an (m, n) -type. Suppose $\Theta = (T, E)$. Let $S \subseteq T$ be the image of ι . Then we define $\text{Cl}(\mathbb{T})$ to be the graph with vertex set T and with edge set $E \cup \binom{S}{2}$.

Recall that a graph Γ is called k -decomposable if there exists a k -element set of vertices whose deletion makes the graph disconnected. Moreover, Γ is called $(k+1)$ -connected if it is l -indecomposable, for all $l \in \{0, \dots, k\}$.

Lemma 24. *Let $m+2 \leq n$. An (m, n) -type $\mathbb{T} = (\Delta, \iota, \Theta)$ is (m, n) -irreducible if and only if $\text{Cl}(\mathbb{T})$ is $(m+1)$ -connected.*

Proof. “ \Rightarrow ” Suppose that \mathbb{T} is (m, n) -reducible. Then $\mathbb{T} \cong \mathbb{T}_1 \oplus_e \mathbb{T}_2$ for some graph-types $\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$ and $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$ such that the order of Δ_2 is $k \leq m$, $\mathbb{T}_1 \not\cong \mathbb{T}$ and $\mathbb{T}_2 \not\cong \mathbb{T}$, and where $e : \Delta_2 \hookrightarrow \Theta_1$ is an embedding. By the definition of $\mathbb{T}_1 \oplus_e \mathbb{T}_2$ there exist embeddings

$\lambda_1 : \Theta_1 \hookrightarrow \Theta$ and $\lambda_2 : \Theta_2 \hookrightarrow \Theta$, such that the following is a pushout-square:

$$\begin{array}{ccc} \Theta_2 & \xrightarrow{\lambda_2} & \Theta \\ \iota_2 \uparrow & & \uparrow \lambda_1 \\ \Delta_2 & \xrightarrow{e} & \Theta_1. \end{array}$$

Let M_1 be the image of λ_1 in Θ , M_2 be the image of λ_2 in Θ , and let M_3 be the image of $\lambda_1 \circ e$ in Θ . Let $N_1 := M_1 \setminus M_3$, $N_2 := M_2 \setminus M_3$, $N_3 := M_3$. Then, by the construction of pushouts of graphs, we have that $V(\Theta)$ is a disjoint union of N_1 , N_2 , and N_3 . Moreover, we have that there are no edges between vertices from N_1 and N_2 in Θ . Let us now observe that N_1 , N_2 , N_3 are non-empty sets. Indeed, suppose that N_1 is empty. Then, since $M_3 \subseteq M_1$, we have $M_3 = M_1$. It follows that e is an isomorphism. Hence also λ_2 is an isomorphism. Since the order of Δ_1 is equal to m , we obtain that $|M_3| \geq m$. On the other hand, $|M_3| = k \leq m$. Thus, $|M_3| = m$. It follows that ι_1 is an isomorphism, too. But now we get that $\mathbb{T}_2 \cong \mathbb{T}$ — contradiction. Thus, $N_1 \neq \emptyset$.

On the other hand, if N_2 is empty, then we obtain immediately that $\mathbb{T}_1 \cong \mathbb{T}$ — contradiction. Thus $N_2 \neq \emptyset$.

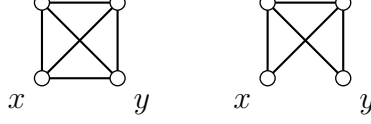
Finally, N_3 is non-empty, by construction.

Now we observe that in $\text{Cl}(\mathbb{T})$ there are still no edges between vertices from N_1 and vertices from N_2 . Thus, removing all vertices from N_3 from $\text{Cl}(\mathbb{T})$ makes the remainder disconnected. It follows that $\text{Cl}(\mathbb{T})$ is k -decomposable. Consequently, \mathbb{T} is not $(m+1)$ -connected.

“ \Leftarrow .” Suppose that $\hat{\Theta} := \text{Cl}(\mathbb{T})$ is not $(m+1)$ -connected. Then there exists some $k \leq m$ such that $\hat{\Theta}$ is k -decomposable. Thus, there exists subsets N_1, N_2, N_3 of $V(\hat{\Theta})$, such that $N_1 \cup N_2 \cup N_3 = V(\hat{\Theta})$, $N_1, N_2 \neq \emptyset$, $|N_3| = k$, and such that there are no edges in $\hat{\Theta}$ between vertices from N_1 and vertices from N_2 . Thus, if $M \subseteq V(\Theta)$ denotes the image of ι , then we have $M \subseteq N_1 \cup N_3$ or $M \subseteq N_2 \cup N_3$. Without loss of generality, assume that $M \subseteq N_1 \cup N_3$. Let Δ_1 be the subgraph of Θ induced by M . Let Θ_1 be the subgraph of Θ induced by $N_1 \cup N_3$. Let Δ_2 be the subgraph of Θ induced by N_3 , and let Θ_2 be the subgraph of Θ induced by $N_2 \cup N_3$. Finally, let $e : \Delta_2 \hookrightarrow \Theta_1$ be the identical embedding. Then, with $\mathbb{T}_1 = (\Delta_1, \iota_1, \Theta_1)$ and $\mathbb{T}_2 = (\Delta_2, \iota_2, \Theta_2)$ (where ι_1 and ι_2 are identical embeddings), we obtain $\mathbb{T} \cong \mathbb{T}_1 \oplus_e \mathbb{T}_2$. By construction we have that \mathbb{T}_1 is of order (m, l_1) where $l_1 = |N_1 \cup N_3| < n$ and that \mathbb{T}_2 is of order (k, l_2) , where $l_2 = |N_2 \cup N_3| < n$. Thus neither \mathbb{T}_1 , nor \mathbb{T}_2 is isomorphic to \mathbb{T} . Consequently, \mathbb{T} is (m, n) -reducible. \square

Corollary 25. *Let Γ be a graph that satisfies the t -vertex condition for $t \geq 3$. Then, in order to verify the $(t+1)$ vertex condition it suffices to test the \mathbb{T} -regularity for all those graph-types \mathbb{T} of order $(2, t+1)$ for which $\text{Cl}(\mathbb{T})$ is 3-connected.*

Example 26. The only 3-connected graph of order 4 is the complete graph K_4 . Thus, the only $(2, 4)$ -irreducible $(2, 4)$ -types are:



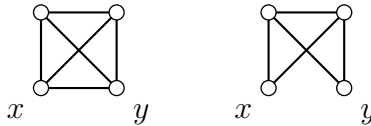
Definition 27. Let Γ be a graph and let $v \in V(\Gamma)$. Then with $\Gamma_1(v)$ we will denote the subgraph of Γ induced by the neighbors of v . Moreover, with $\Gamma_2(v)$ we denote the subgraph of Γ induced by the non-neighbors of v (except v itself). $\Gamma_1(v)$ and $\Gamma_2(v)$ are called the first and the second subconstituent of Γ with respect to v , respectively.

Proposition 28. Let Γ be an (m, n) -regular graph where $m \geq 1$, and let $v \in V(\Gamma)$. Then $\Gamma_1(v)$ and $\Gamma_2(v)$ are both $(m-1, n-1)$ -regular. \square

4. CHECKING THE t -VERTEX CONDITION

It is clear, that every graph satisfies the 1-vertex condition. Moreover, a graph satisfies the 2-vertex condition if and only if it is regular. A bit less obvious but rather straight forward is the observation that a graph satisfies the 3-vertex condition if and only if it is strongly regular — i.e. it is regular and the number of joint neighbors of every edge is equal to a constant λ and the number of joint neighbors of every non-edge is equal to a constant μ . A criterion for the 4-vertex condition is given by:

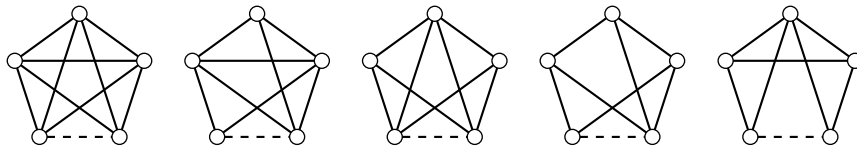
Theorem 29 (M.D. Hestenes, D.G. Higman [9]). *Let Γ be a strongly regular graph. Then, in order to verify the 4-vertex condition it suffices to test the \mathbb{T} -regularity for the following two graph-types of order $(2, 4)$:*



In our terminology, this is a special case of Corollary 25.

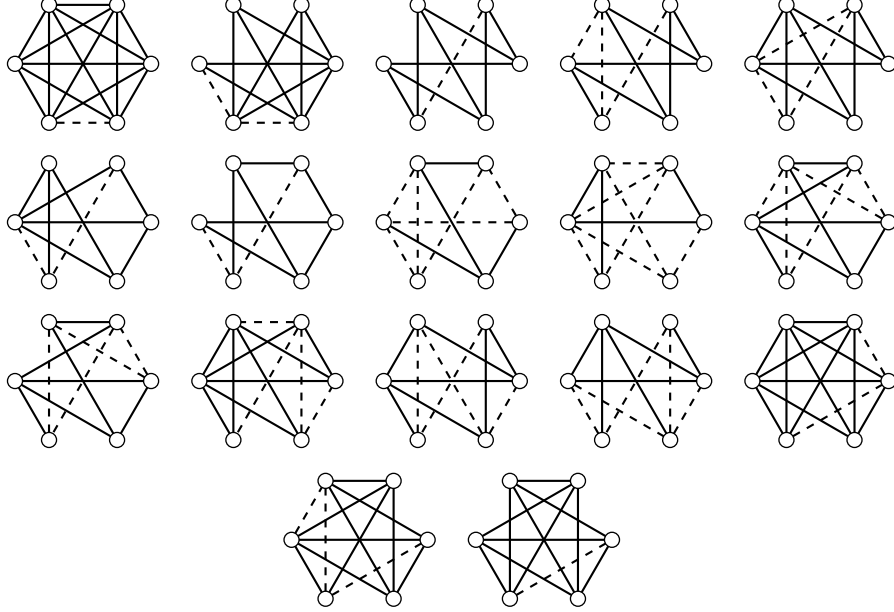
In [19, Thm. 4.9] S. Reichard proved that a graph satisfying the 4 vertex condition will satisfy the 5-vertex condition if and only if it is regular for a list of 16 graph-type. The following proposition reduces the number of graph-types to be tested to 10:

Proposition 30. *Given a graph Γ that fulfills the 4-vertex condition. Then in order to test whether Γ satisfies also the 5-vertex condition it suffices to count the graph-types in the table below.*



Since the number of unlabeled 3-connected graphs of order 6 is not too large, we give also a necessary and sufficient condition for a graph with the 5-vertex condition to satisfy the 6-vertex condition:

Proposition 31. *Given a graph Γ that fulfills the 5-vertex condition. Then in order to test whether Γ satisfies also the 6-vertex condition it suffices to count the graph-types in the table below.*



5. POINT-GRAPHS OF PARTIAL QUADRANGLES

An incidence structure is a triple $(\mathcal{P}, \mathcal{L}, I)$ Where \mathcal{P} is a set of points (denoted by capital Latin letters P, Q, \dots), \mathcal{L} is a set of lines (denoted by small Latin letters l, s, t, \dots), and $I \subseteq \mathcal{P} \times \mathcal{L}$ is an incidence relation. The elements of I are called *flags* and the elements of $(\mathcal{P} \times \mathcal{L}) \setminus I$ are called *anti-flags*. A point P is called incident with a line l if (P, l) is a flag. Slightly abusing the notation we write in this case $P \in l$. Two distinct flags (P, p) and (Q, q) are called *collinear* if $p = q$, and *concurrent* if $P = Q$. Two distinct points P and Q are called collinear if there exists a line l such that (P, l) and (Q, l) are flags. In this case we say that l goes through P and Q . Dually, we say that two lines p and q are intersection each other if there is a point P such that $P \in p$ and $P \in q$.

For every incidence structure we may define its *point-graph*. This is a simple graph which has as vertices the points of the incidence structure such that between two points there is an edge if and only if the points are collinear.

Following we will restrict our attention to so called partial linear spaces:

A *partial linear space of order (s, t)* (short $\text{PLS}(s, t)$) is an incidence structure $(\mathcal{P}, \mathcal{L}, I)$ with the following properties:

- (PLS1) every line is incident with the same number $s + 1$ of points,
- (PLS2) every point is incident with the same number $t + 1$ of lines,
- (PLS3) through any two distinct points goes at most one line,

If two lines p and q of a partial linear space intersect each other, then we denote the unique point of intersection by $p \cap q$.

We are interested in partial linear spaces because there are interesting classes of partial linear spaces whose point graphs are strongly regular graphs. Two such classes we will define in the sequel:

A *generalized quadrangle of order (s, t)* (short $\text{GQ}(s, t)$) is a partial linear space of order (s, t) with the the following additional property:

- (GQ1) for every anti-flag (P, q) there is a unique point Q such that P and Q are collinear and such that $Q \in q$.

It is well-known that the point-graph of a generalized quadrangle of order (s, t) is strongly regular with parameters (v, k, λ, μ) where

$$v = (s + 1)(st + 1) \quad k = s(t + 1) \quad \lambda = s - 1 \quad \mu = t + 1$$

Axiom GQ1 ensures that a generalized quadrangle does not contain triples of pairwise collinear points that are not all three on one line. From this follows that every set of points that induces a clique in the point-graph, is a subset of some line. In particular the generalized quadrangle can be reconstructed from its point-graph up to isomorphism by taking as points the vertices of the point-graph, as lines the maximal cliques and as incidence-relation the \in -relation. Moreover, the point-graph of a generalized quadrangle can not contain $K_4 - e$ as induced subgraph because this would imply the existence of two maximal cliques that intersect in at least two points which can not happen because of axiom PLS3.

In [2] P. J. Cameron examined point-graphs of generalized quadrangles and made the above observations. These observation lead him to study strongly regular graphs that do not contain $K_4 - e$. It turns out that such graphs always arise as point-graph of certain partial linear spaces. The class of partial linear spaces that have as a point-graph an srg without $K_4 - e$ as induced subgraph, are called *partial quadrangles*. Following we give an axiomatization:

A *partial quadrangle* partial with parameters (s, t, μ) (short $\text{PQ}(s, t, \mu)$) is a partial linear space $(\mathcal{P}, \mathcal{L}, I)$ of order (s, t) with the following properties:

- (PQ1) if three points are pairwise collinear, then they are all three on one line,
- (PQ2) for every pair (P, Q) of non-collinear points there exist μ points X that are collinear with both points P and Q .

Clearly, every generalized quadrangle is a partial quadrangle. The inclusion of these two classes is proper as, e.g. srgs with parameters (v, k, λ, μ) where $\lambda \in \{0, 1\}$ define partial quadrangles.

Theorem 32 (P. J. Cameron [2, Thm. 2]). *Let $\Gamma = (V, E)$ be a strongly regular graph with parameters (v, k, λ, μ) . Then Γ is isomorphic to the point-graph of a partial quadrangle if and only if $\mu > 0$ and it does not contain any induced subgraph isomorphic to $K_4 - e$. \square*

Let us recall that starting from a strongly regular graph Γ with parameters (v, k, λ, μ) that has no induced subgraph isomorphic to $K_4 - e$, we can construct a partial quadrangle by taking as points the vertices of Γ and as lines the maximal cliques. The resulting partial quadrangle has parameters $(\lambda + 1, \frac{k}{\lambda + 1} - 1, \mu)$.

Now we are ready to formulate the first result of this section:

Theorem 33. *Let Γ be a strongly regular graph that does not contain $K_4 - e$ as an induced subgraph. Then Γ satisfies the 5-vertex condition.*

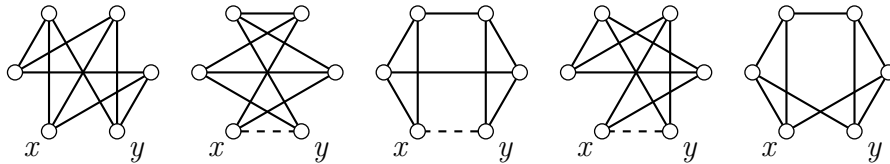
Proof. As first we note that by Theorem 29, in order to test the 4-vertex condition for Γ it is enough to test it for K_4 (the other type can not occur as induced subgraph in Γ).

Secondly we note that from all graph-types given in Proposition 30 only K_5 does not contain $K_4 - e$ as an induced subgraph. Thus, in order to test the 5-vertex condition, we have only to count 4-cliques and 5-cliques containing a fixed edge, respectively.

However, for a fixed edge $\{x, y\}$ the number of 4-cliques and 5-cliques containing this edge is equal to $\binom{\lambda}{2}$ and $\binom{\lambda}{3}$, respectively. \square

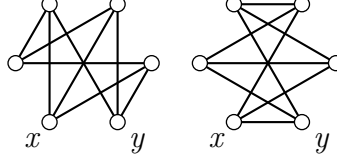
If we want to test whether the point-graph of a partial quadrangle also fulfills the 6-vertex condition, the list of graph-types from Theorem 31 reduces drastically:

Proposition 34. *Let Γ be the point-graph of a partial quadrangle. Then in order to test the 6-vertex condition for Γ it suffices to check it for the following 8 graph-types:*



In the special case of generalized quadrangles S. Reichard showed in [19] is showed that among these 8 graph types there are 6 types \mathbb{T} such that the point graph of every generalized quadrangle \mathbb{T} -regular. Together with this observation we obtain:

Proposition 35. *The point graph of a generalized quadrangle satisfies the 6-vertex condition if and only if it is regular for the following graph-types:*



6. (3, 7)-REGULAR GRAPHS

Recall, that in a partial linear space, three pairwise no-collinear points are called a *triad*. Moreover, a *center* of a triad is a point collinear to all three points of the triad.

Theorem 36 (P. J. Cameron [2, Thm. 2]). *Given a partial quadrangle of order \$(s, t, \mu)\$. Then*

$$(5) \quad \left(s(t-1) + (\mu-1)(\mu-2) \right) \frac{(t+1)ts^2}{\mu-1-(t+1)s+\mu} \geq \mu(t-1)^2s^2.$$

Moreover, equality holds if and only if every triad in the partial quadrangle has the same number \$c\$ of centers. In this case we have

$$c = 1 + \frac{(\mu-1)(\mu-2)}{s(t-1)}$$

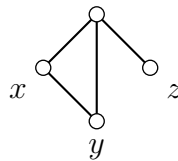
For the special case of generalized quadrangles this simplifies to the following well-known result by Higman:

Theorem 37 (D. G. Higman 1971 cf. [2, Cor. to Thm.1]). *Given a generalized quadrangle of order \$(s, t)\$. Then \$s^2 \geq t\$. Moreover, equality holds if and only if every triad in the generalized quadrangle has the same number \$(s+1)\$ of centers.*

Corollary 38 ([20, Cor.5.1]). *Let \$\Gamma\$ be the point-graph of a generalized quadrangle of order \$(q, q^2)\$. Then \$\Gamma\$ is 3-isoregular.*

Proposition 39. *Let \$\Pi\$ be a partial quadrangle of order \$(s, t, \mu)\$, such that every triad in \$\Pi\$ has the same number \$c\$ of centers, and let \$\Gamma\$ be its point graph. Then \$\Gamma\$ is 3-isoregular if and only if either \$\Pi\$ is a generalized quadrangle and \$t = s^2\$, or \$\Gamma\$ is triangle-free (i.e., \$s = 1\$).*

Proof. “ \Rightarrow ” Suppose that \$\Gamma\$ is 3-isoregular. Consider the following graph-type \$\mathbb{T} = (\Delta, \iota, \Theta)\$ of order \$(3, 4)\$:

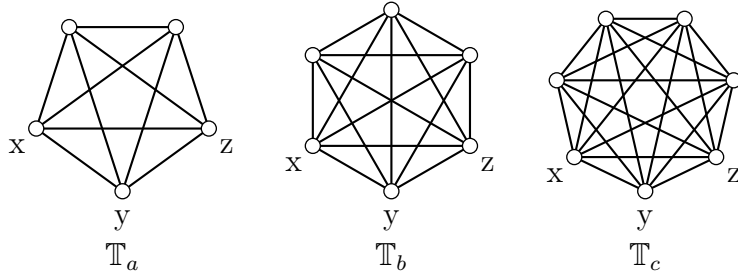


Then any embedding κ of Δ into Γ determines a line l of Π (spanned by $\kappa(x)$ and $\kappa(y)$) and a vertex $p = \kappa(z)$ not on this line such that neither $\kappa(x)$ nor $\kappa(y)$ is collinear with p . In any partial quadrangle there exists at most one vertex q on l that is collinear with p (otherwise Π would contain a triangle of lines). So we have $\#(\Gamma, \mathbb{T}) \in \{0, 1\}$. If $\#(\Gamma, \mathbb{T}) = 0$, then Γ is triangle-free and if $\#(\Gamma, \mathbb{T}) = 1$, then Π is a generalized quadrangle. By Theorem 37, we obtain that $t = s^2$.

“ \Leftarrow .” If Π is a generalized quadrangle, then Γ is 3-isoregular, by Corollary 38. So suppose that Γ is triangle-free. Let u, v, w be three mutually distinct vertices of Γ . If the subgraph of Γ induced by u, v , and w contains an edge, then they have no common neighbors (otherwise Γ would contain triangles). So u, v, w form a triad in Π . Hence, they have c common neighbors in Γ . Consequently, Γ is 3-isoregular. \square

Theorem 40. *Let Γ be the point-graph of a partial quadrangle and suppose that Γ is 3-isoregular. Then Γ is $(3, 7)$ -regular.*

Proof. By Proposition 39, Γ is 3-isoregular. Hence it is $(3, 4)$ -regular. By Lemma 22, in order to prove $(3, 5)$ -regularity of Γ it suffices to prove the \mathbb{T} -regularity of Γ for all $(3, 5)$ -irreducible graph-types. By Lemma 24, \mathbb{T} is $(3, 5)$ -irreducible if and only if $\text{Cl}(\mathbb{T})$ is 4-connected. Since Γ does not have $K_4 - e$ as an induced subgraph, it suffices to prove \mathbb{T} -regularity for types $\mathbb{T} = (\Delta, \iota, \Theta)$ where Θ does not contain an induced subgraph isomorphic to $K_4 - e$. A computer search reveals that there is just one type \mathbb{T}_a that fulfills all these conditions:



However, it is easy to see that

$$\#(\Gamma, \mathbb{T}) = \begin{cases} \binom{s-2}{2} & s \geq 2 \\ 0 & \text{else.} \end{cases}$$

Thus, Γ is $(3, 5)$ -regular.

With the same reasoning as before and again using a computer, we obtain that Γ is $(3, 6)$ -regular if and only if it is \mathbb{T}_b -regular. However, it is easy to see that

$$\#(\Gamma, \mathbb{T}) = \begin{cases} \binom{s-2}{3} & s \geq 2 \\ 0 & \text{else.} \end{cases}$$

Thus, Γ is $(3, 6)$ -regular.

Finally, once more using the same reasoning as above and using a computer, we obtain that Γ is $(3, 7)$ -regular if and only if it is \mathbb{T}_c -regular. However, it is easy to see that

$$\#(\Gamma, \mathbb{T}) = \begin{cases} \binom{s-2}{4} & s \geq 2 \\ 0 & \text{else.} \end{cases}$$

Thus, Γ is $(3, 7)$ -regular. \square

The previous theorem generalizes a result by Reichard ([20, Thm.1]) that states that the point-graphs of generalized quadrangles of order (q, q^2) satisfy the 7-vertex condition.

Corollary 41. *Let Γ be the point graph of a partial quadrangle and suppose that Γ is 3-isoregular. Then, for every $v \in V(\Gamma)$, the second subconstituent $\Gamma_2(v)$ satisfies the 6-vertex condition.*

The previous corollary applies in particular to the point graphs of generalized quadrangles of order (q, q^2) . It is clear, that if Γ is the point graph of a generalized quadrangle of order (q, q^2) , then for every $v \in V(\Gamma)$, the graph $\Gamma_2(v)$ is the point graph of a partial quadrangle of order $(q-1, q^2, q^2-q)$. On the other hand it was shown by A. A. Ivanov and S. V. Shpektorov in [10, Thm.A] that every partial quadrangle of order $(q-1, q^2, q^2+q)$ extends to a generalized quadrangle of order (q, q^2) . Consequently, we obtain.

Corollary 42. *Let Γ be the point graph of a partial quadrangle of order $(q-1, q^2, q^2-q)$. Then Γ satisfies the 6-vertex condition.*

Example 43. There exist infinite family of generalized quadrangles of order (q, q^2) whose point graphs are non-rank-3-graphs (cf. [12, 13, 17]). By Theorem 40, the point graph of any such generalized quadrangle is $(3, 7)$ -regular. The second subconstituents of these graphs give rise to a hitherto unknown family of non-rank-3-graphs satisfying the 6-vertex condition.

The smallest actual example is the point graph Γ of a non-classical generalized quadrangle of order $(5, 25)$. It has parameters $(v, k, \lambda, \mu) = (756, 130, 4, 26)$. Its automorphism group is intransitive of rank 11.

Γ has two non-isomorphic second subconstituents Γ' and Γ'' . Both satisfy the 6-vertex condition and both are in turn point graphs of partial quadrangles of order $(4, 25, 20)$. The Automorphism group of Γ' is intransitive of rank 52 and the automorphism group of Γ'' is transitive of rank 5.

Proposition 44. *Let Γ be the point graph of a partial quadrangle, and suppose that Γ is 3-isoregular. Then Γ satisfies the 8-vertex condition if and only if it is regular for $\mathbb{T}_{1,1}, \mathbb{T}_{2,1}, \mathbb{T}_{2,2}, \mathbb{T}_{3,1}, \dots, \mathbb{T}_{3,4}$ from Figure 1 (for better visibility, $\mathbb{T}_{3,1}, \dots, \mathbb{T}_{3,4}$ are given as geometric configurations).*

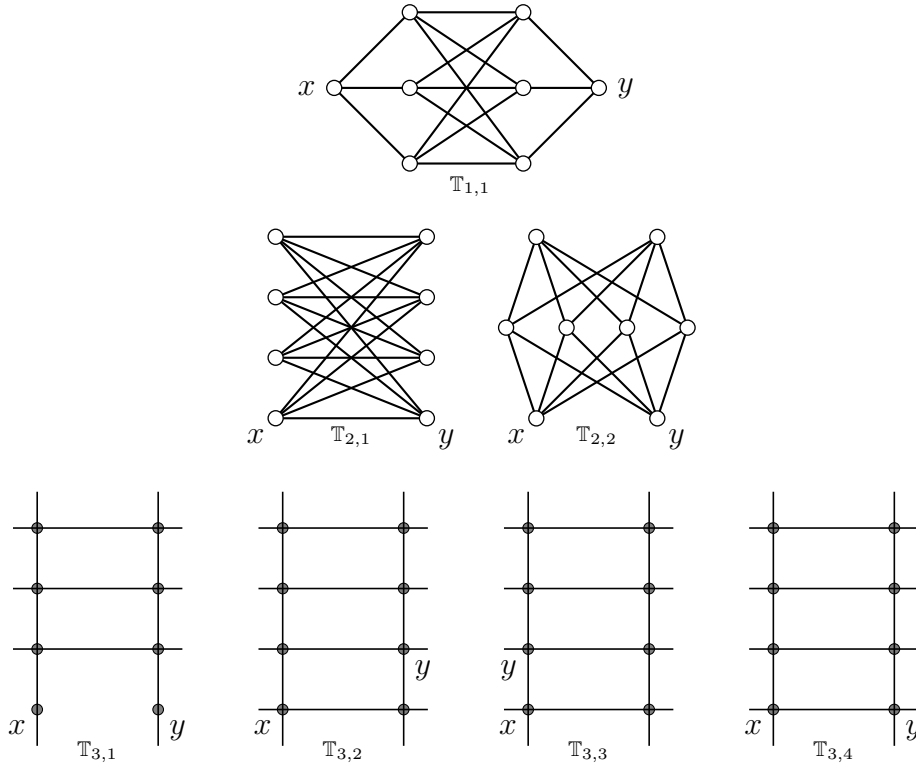


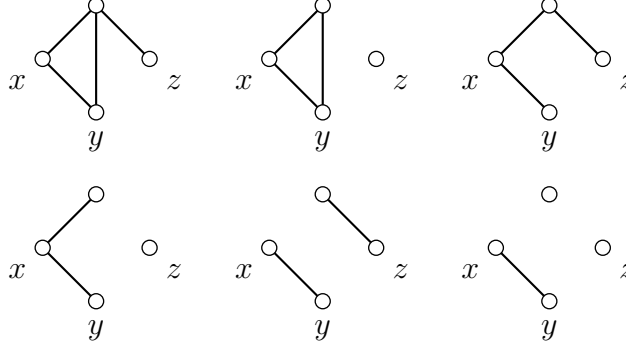
FIGURE 1. Graph types relevant for the 8-vertex condition in point graphs of partial and generalized quadrangles

Proposition 45. *The point graph of a generalized quadrangle of order (q, q^2) satisfies the 8-vertex condition if and only if it is regular for $\mathbb{T}_{1,1}$, $\mathbb{T}_{2,1}$, and $\mathbb{T}_{2,2}$ from Figure 1.*

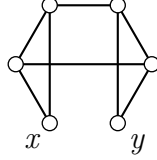
Let us at the end have a look on partial quadrangles Π in which every triad has the same number c of centers, but where the point-graph Γ is not necessarily 3-isoregular.

Lemma 46. *Let Π be a partial quadrangle in which every triad has c centers, and let Γ be the point graph of Π . Then Γ is regular for all*

graph-types of order $(3, 4)$, except possibly the following:



Proposition 47. *Let Π be a partial quadrangle in which every triad has c centers, and let Γ be the point graph of Π . Then Γ satisfies the 6-vertex condition if and only if it is regular for the following graph-type:*



7. CONCLUDING REMARKS

The (m, n) -regularity, introduced in Section 3 is a very strong condition. It is in fact interesting only for $m \leq 4$, because any 5-isoregular graph is 5-homogeneous and, in fact, homogeneous [8, 3, 7]. At the first sight this appears to limit the use of the general theory of regularity conditions that is developed in this paper. However, in principle the definitions and results from Section 3 are applicable to other categories of combinatorial objects. To mind spring finite metric spaces (possibly with integer or with rational distances), directed graphs, or semilinear spaces.

For the category of finite graphs, the most interesting are (m, n) -regular graphs where $m \in \{2, 3, 4\}$. Here the goal is to find (m, n) -regular graphs that are not m -homogeneous and, if feasible, to classify such graphs completely, up to isomorphism.

The $(2, t)$ -regular graphs correspond exactly to the graphs that satisfy the t -vertex condition. There is a long-standing conjecture by M. Klin [5], that there exists a natural number t_0 such that for each $t \geq t_0$ all $(2, t)$ -regular graphs are 2-homogeneous (i.e., they are rank 3 graphs). The largest t for which the existence of a non rank 3, $(2, t)$ -regular graph is settled is $t = 7$, due to [20, Thm.2]. Thus in Klin's conjecture we have $t_0 \geq 8$.

$(3, t)$ -regular graphs, up till now, were known only for $t = 4$ (obviously, apart from the 3-homogeneous graphs). These are known as 3-isoregular graphs or as triply regular graphs. In this paper, the first

cases of non 3-homogeneous $(3, 7)$ -regular graphs are observed. Among the examples there are graphs whose automorphism group is intransitive. In view of Klin's conjecture and in view that $(3, t)$ -regular graphs appear to be much rarer than $(2, t)$ -regular graphs, it seems sensible to conjecture that there exists a t_0 such that all $(3, t)$ -regular graphs with $t \geq t_0$ are 3-homogeneous. This paper shows that if such t_0 exists, then $t_0 \geq 8$. Note that every $(3, t)$ -regular graph is $(2, t)$ -regular. Thus, if Klin's conjecture turns out to be true for some t_0 , then the latter conjecture can be answered using the classification of rank 3 graphs.

There is known only one $(4, 5)$ -regular graph that is not 4-homogeneous—the McLaughlin graph on 275 vertices. A computer experiment showed that this graph is not $(4, 6)$ -regular.

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